SPECIAL FORMS OF THE MOMENTAL ELLIPSOID OF A BODY.

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In works on analytical mechanics it is shown

1. That for every point within or without a body an ellipsoid may be constructed such that the square of the reciprocal of any semidiameter is equal to the moment of inertia of the body with respect to an axis coincident with this semidiameter. This ellipsoid is known as a momental ellipsoid.

2. That if H' be the moment of inertia of a body with respect to any axis in space, H its moment of inertia with respect to a parallel axis through the centre of mass of the body, p the perpendicular distance between these two axes, M the mass of the body; then

$$H' = H + Mp^2.$$

These two propositions will be used here to discover whether there are any points for which: (a) the momental ellipsoid has its two circular sections at right angles to each other; (b) the momental ellipsoid is a sphere. We are guided in the search by the fact that the momental ellipsoid of every point situated on a normal to a circular section of the central ellipsoid (the momental ellipsoid which has the centre of mass of the body for its centre) has one circular section perpendicular to the normal upon which the point is situated, and we are to find whether for any particular point or points on this normal the other circular section is perpendicular to this one.

Let $g \equiv$ the centre of mass of the body,

$M \equiv$ the mass of the body,

$O \equiv$ one of the points sought on the normal to a circular section of the central ellipsoid,

$d \equiv$ the distance from g to O,

$r \equiv$ any semidiameter of the ellipsoid whose centre is O,

$\rho \equiv$ a semidiameter parallel to r, of the ellipsoid whose centre is g,

$a > b > c \equiv$ the semiaxes of the ellipsoid whose centre is g,

$\alpha \equiv$ the angle between r, or ρ, and the line d,

$\beta \equiv$ the angle between d and the semiaxis c,

$s \equiv$ the semidiameter of the central ellipsoid coincident with d.

The normal to a circular section of the central ellipsoid (at its centre) evidently lies in the plane ac. The angle β is also the angle which the circular sections of the central ellipsoid make with the axis a.

We readily find
$$\cos^2\beta = \frac{a^2(b^2 - c^2)}{b^2(a^2 - c^2)}, \tag{1}$$

or
$$\cos 2\beta = \frac{a^2(b^2 - c^2) - c^2(a^2 - b^2)}{b^2(a^2 - c^2)}; \tag{2}$$

$$\frac{1}{s^2} = \frac{\cos^2\beta}{c^2} + \frac{\sin^2\beta}{a^2}. \tag{3}$$

Propositions 1 and 2 give
$$\frac{1}{r^2} = \frac{1}{\rho^2} + Md^2\sin^2\alpha. \tag{4}$$

As already pointed out, the section perpendicular to d, of the ellipsoid whose centre is O, is circular for all values of d. Examine now the section containing the line d and normal to the plane ac.

If $\alpha = 0$, then by hypothesis $\rho = s$, and by (4)
$$\frac{1}{r^2} = \frac{1}{\rho^2} = \frac{1}{s^2};$$
$$\therefore r = s.$$

If $\alpha = \frac{1}{2}\pi$, then in like manner $\rho = b$, and by (4)
$$\frac{1}{r^2} = \frac{1}{b^2} + Md^2. \tag{5}$$

If the section is circular, r in this equation must equal s, the value found for r when $\alpha = 0$. That is, d is determined by the condition
$$\frac{1}{s^2} = \frac{1}{b^2} + Md^2,$$

or
$$Md^2 = \frac{1}{s^2} - \frac{1}{b^2}, \tag{6}$$

which (1) and (3) reduce to
$$Md^2 = \frac{1}{a^2} + \frac{1}{c^2} - \frac{2}{b^2}. \tag{7}$$

Remembering that for the section under consideration ρ is the semidiameter of an ellipse whose axes are s and b, and makes an angle α with s, we see that
$$\frac{1}{\rho^2} = \frac{\cos^2\alpha}{s^2} + \frac{\sin^2\alpha}{b^2}. \tag{8}$$

The substitution in (4) of the values of Md^2 and $1/\rho^2$ obtained from equations (6) and (8) gives
$$\frac{1}{r^2} = \frac{1}{s^2};$$

that is, $r = s$ for all values of a. Hence this section is circular, and at right angles to the circular section which is normal to d.

Equation (7) gives two values for d, equal with opposite signs, which shows there are two points, on opposite sides of g, satisfying the imposed condition. There are two points on the normal to the other circular section of the central ellipsoid; hence there are, in all, four points where the two circular sections of the momental ellipsoid are at right angles to each other.

We see from equation (7) that for d to have real values we must have

$$\frac{1}{a^2} + \frac{1}{c^2} > \frac{2}{b^2},$$

which means that β must be less than $45°$.

If
$$\frac{1}{a^2} + \frac{1}{c^2} = \frac{2}{b^2}, \quad \text{or} \quad \beta = 45°,$$
$$d = 0,$$

and the four points reduce to one at the centre of the central ellipsoid. The central ellipsoid itself now has its two circular sections at right angles to each other. If a, c, and M are constant, and b variable, which would make β variable, the four points would shift their positions, and their locus is easily found to be the lemniscate of Bernoulli cutting the axis c at right angles. The equation of the lemniscate is

$$d^2 = \frac{a^2 - c^2}{Ma^2c^2} \cos 2\beta.$$

If
$$\frac{1}{a^2} + \frac{1}{c^2} < \frac{2}{b^2},$$

d in equation (7) is imaginary, and there is no point on the normal to the circular sections of the central ellipsoid for which the momental ellipsoid has its two circular sections at right angles to each other.

To find whether the points found above are centres of *ellipsoids* as distinguished from *spheres*, examine a third section, the one made by the plane ac.

ρ makes an angle $a + \beta$ with c, and lies in the plane ac; therefore, as in (3),

$$\frac{1}{\rho^2} = \frac{\cos^2(a+\beta)}{c^2} + \frac{\sin^2(a+\beta)}{a^2}.$$

The substitution in (4) of the values of $1/\rho^2$ and Md^2 obtained from this and equation (7) gives

$$\frac{1}{r^2} = \frac{a^2b^2\cos^2(a+\beta) + b^2c^2\sin^2(a+\beta) + [a^2(b^2-c^2) - c^2(a^2-b^2)]\sin^2 a}{a^2b^2c^2}. \quad (9)$$

Pass to rectangular co-ordinates with O as a centre, the line d as axis of z, and a line perpendicular to d as axis of x; and we find

$$\begin{aligned} & z^2 [a^2b^2 \cos^2 \beta + b^2c^2 \sin^2 \beta] \\ + x^2 [a^2b^2 \sin^2 \beta + b^2c^2 \cos^2 \beta + a^2 (b^2 - c^2) - c^2 (a^2 - b^2)] \\ & - zxb^2 (a^2 - c^2) \sin 2\beta = a^2b^2c^2. \end{aligned} \tag{10}$$

This is the equation of an ellipse. Its principal axes are found to be inclined 45° to z or d. Let c_1 and a_1 be its semiaxes; then

$$c_1 = \frac{abc}{\sqrt{[a^2b^2 + b^2c^2 - a^2c^2 + b^2(a^2 - c^2)\sin\beta\cos\beta]}}, \tag{11}$$

$$a_1 = \frac{abc}{\sqrt{[a^2b^2 + b^2c^2 - a^2c^2 - b^2(a^2 - c^2)\sin\beta\cos\beta]}}. \tag{12}$$

We see that this section of the ellipsoid whose centre is O is an ellipse, and hence the ellipsoid is not a sphere.

To find for what points the ellipsoids are spheres we make $c_1 = a_1$ in equations (11) and (12). This requires

$$\beta = 0;$$
$$\therefore b = a;$$
$$\therefore c_1 = a_1 = c.$$

Hence the points sought lie on c; there are evidently two such points, one on each side of the centre g. Equation (7) now reduces to

$$\begin{aligned} Md^2 &= \frac{1}{a^2} + \frac{1}{c^2} - \frac{2}{a^2} \\ &= \frac{1}{c^2} - \frac{1}{a^2}; \end{aligned}$$

and in order that d may have a real value, c must be less than a. We conclude, therefore, that if the central ellipsoid be one of revolution about its shortest axis, there are two points on that axis for which the momental ellipsoids are spheres.

The result just reached concerning the sphere may be obtained very simply by analytical methods as in Bartlett's Analytical Mechanics, p. 169. Poinsot in his *Théorie Nouvelle de la Rotation des Corps*, p. 63, gives a simple proof that for a sphere O must be on c, and $c < a = b$; but his proof is not conclusive. In substance, he says that since by hypothesis the ellipsoid centre O is a sphere, every section of it is circular; therefore, from the relation

$$H' = H + Md^2,$$

the section of the central ellipsoid by a plane normal to the line joining its centre

with O, must be circular; therefore, the central ellipsoid must be one of revolution. The last is a *non-sequitur*, because, as we have seen, if O is on the normal to a circular section, it satisfies the condition that the two parallel sections normal to d, one with centre at O and the other with centre at g, the centre of mass of the body, shall both be circular. Poinsot's reasoning shows no more than that O must be on a normal to a circular section of the central ellipsoid; it does not show that this circular section must be a principal section, although he gives that as his conclusion.

It must be confessed that, while it was shown above that under certain conditions there are four points for which the momental ellipsoid has two circular sections at right angles to each other, the analysis does not show conclusively that there are *only* four such points. We have discovered all such points as are situated on normals to central circular sections, but are there no such points elsewhere?

ON THE MASS OF TITAN.

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There is a numerical error in the value of a_6 as given on page 169, Vol. III of this journal. The correct value is approximately $-\frac{1}{2}a_3^2$. The introduction of this in equation (19), p. 166, adds other appreciable terms, the largest of which is $-3a_3a_6 = \frac{3}{2}a_3^3$. If, also, we take into account the term $m'P_0$ in ν, we may, in the place of equation (31), write

$$(2a_3P_0 + P_3)\,m' = \left[1 - 9\left(\frac{n' - n}{n}\right)^2\right] a_3,$$

approximately; whence

$$m' = \frac{1}{4617}.$$

This change in the mass leads, of course, to corresponding changes in the values of the coefficients a_1, n_1, etc.